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A Characterization of Best Approximations with Restricted Ranges

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Restricted range approximation in uniform norm from an extended Haar space of a certain order is an important and widely applicable problem in restricted approximations. In the past 20 years or more, many authors have investigated the characterization of best approximations in various special cases of restricted range approximation, which include approximation with interpolatory constraints, one-sided approximation, and copositive approximation. But the characterization in the general case is still an open question. The paper gives a general characterization theorem in the form of convex hull and alternation. Many known important results are exactly its special cases.

1. INTRODUCTION

Let $[a, b]$ be a finite interval and $\mathscr{X} \subset [a, b]$ a compact set containing at least $n+1$ points. By $C(\mathscr{X})$ we denote the normed linear space consisting of all the continuous real valued functions defined on $\mathscr{X}$, with the uniform norm $\|\cdot\|$. If $\Phi_n = \text{span}(\varphi_1, ..., \varphi_n)$ is an n-dimensional extended Haar subspace of order r $(1 \leqslant r \leqslant n)$ on $[a, b]$, that is, $\{\varphi_1, ..., \varphi_n\} \subset C([a, b])$ is an extended Chebyshev system of order r on $[a, b]$ (see the definition in [1], Chap. 1, Sect. 2), then we call

$$K = \{q \in \Phi_n : l(x) \leqslant q(x) \leqslant u(x), x \in [a, b]\}$$

the set of generalized polynomials having restricted ranges, where l and u are extended real valued functions defined on $[a, b]$ satisfying $-\infty \leqslant l(x) \leqslant u(x) \leqslant +\infty$. Given $f \in C(\mathscr{X}) \backslash K$, the problem of approximating f by K is important and widely applicable because many standard restricted approximations investigated by many authors are special cases of it. Indeed, if we set $l(x)$ and $u(x)$ properly, we may get interpolatory constrained approximation which has been studied by, for example,

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F. Deutsch [2], one-sided approximation (see [3, Chap. 3, Sect. 8]) containing a special case of positive approximation, copositive approximation which has been studied in [4–6], etc.

On the characterization of best approximations with restricted ranges, G. D. Taylor [7] gave in 1969 a theorem in the form of convex hull and alternation under a hypothesis of $l(x) < u(x)$, $x \in [a, b]$, and a certain continuous condition on l and u. And his investigation in [8] allows $l(x_i) = u(x_i)$ at some nodes x_i, but it is required that l and u have special local Taylor expansions in a neighbourhood of each x_i. Latterly, W. Sippel [9] also considered the equality case with the assumptions that l and u are continuous on $[a, b]$ and have continuous derivatives of sufficiently high order in a neighbourhood of each x_i, as well as some other constraints on f. Clearly, the condition in [7–9] are so strong that in general the results are not applicable to many standard constraints including approximation with interpolatory constraints, one-sided approximation (by the set $\{q \in \Phi_n: q(x) \geqslant l(x)\}$ with l not necessarily continuous), and copositive approximation. Moreover, in 1980, Y. K. Shih [10] investigated the problem with a different assumption that $l(x) + d \leqslant l(x_i) \leqslant u(x) - d \; (d \geqslant 0)$ in a certain deleted neighbourhood of each node x_i. This is still a special case of approximation by K which cannot contain the results in [8] and [9], and cannot be applied to copositive approximation.

In brief, though progress has been achieved in approximation with restricted ranges in the past 20 years or more, the characterization theorem in the general case is still an open question. To solve the problem at last, this paper gives a theorem in the form of convex hull and alternation, which contains all the results in [7–10] on characterization as well as that of [4–6].

2. Notations and Main Results

Given $p \in K$. Since in the case $K = \{p\}$ the problem of characterization is trivial, we always assume that

$$K \setminus \{p\} \neq \varnothing. \tag{1}$$

Based on the definition of the extended Chebyshev system of order r, each $q \in \Phi_n$ has a continuous derivative of order $r-1$. We call $x \in [a, b]$ a zero of order t $(0 \leqslant t \leqslant r)$ of q if $q^{(0)}(x) = \cdots = q^{(t-1)}(x) = 0$, and $q^{(t)}(x) \neq 0$ when $t < r$. It is clear that for $q \not\equiv 0$ there exist at most $n-1$ zeros on $[a, b]$ (counting multiplicities).

We first introduce some needed notations. Let

$$d(p(x), l) = \inf_{\xi \in [a, b]} \sqrt{(\xi - x)^2 + [l(\xi) - p(x)]^2},$$

and define $d(p(x), u)$ similarly. We call

$$X^* = \{x \in [a, b] : d(p(x), l) = d(p(x), u) = 0\}$$

the set of nodes of K. Provided $x \in [a, b)$, by $\tau_{1,1}(x)$ we denote the largest positive integer t subject to

$$\lim_{\xi \to x+0} \frac{u(\xi) - p(\xi)}{|\xi - x|^{t-1}} = 0, \tag{2}$$

and if the above equality holds or does not hold for any positive integer t, then let $\tau_{1,1}(x)$ equal $+\infty$ or 0, respectively. Similarly we define $\tau_{1,-1}(x)$ by the equality

$$\lim_{\xi \to x+0} \frac{p(\xi) - l(\xi)}{|\xi - x|^{t-1}} = 0. \tag{3}$$

Substituting $x - 0$ for $x + 0$ in (2) and (3) we define $\tau_{-1,1}(x)$ and $\tau_{-1,-1}(x)$ for $x \in (a, b]$. Let

$$t_{\mu,\nu}(x) = \begin{cases} \tau_{\mu,\nu}(x) + 1, & \text{if } x \in X^*, \text{ and } \tau_{\mu,\nu}(x) = 0 \text{ or } r, \\ \tau_{\mu,\nu}(x), & \text{otherwise,} \end{cases} \qquad \mu, \nu = \pm 1;$$

$$t_+(x) = \min\{t_{1,1}(x), t_{1,-1}(x)\};$$

$$t_-(x) = \min\{t_{-1,1}(x), t_{-1,-1}(x)\};$$

$$t_\pm(x) = \begin{cases} \max\{t_+(x), t_-(x)\}, & \text{if } x \in (a, b), \\ t_+(x), & \text{if } x = a, \\ t_-(x), & \text{if } x = b; \end{cases}$$

$$\omega = \omega(x) = (-1)^{t_\pm(x)};$$

and

$$t(x) = \begin{cases} t_\pm(x) + 1, & \text{if there exists } \nu \text{ such that} \\ & \qquad t_{1,\nu}(x), t_{-1,-\omega\nu}(x) > t_\pm(x), \\ t_\pm(x), & \text{otherwise.} \end{cases} \tag{4}$$

Then it is easy to check that

$$\begin{cases} t(x) \geqslant 1, & \text{if } x \in X^*, \\ t(x) = 0, & \text{if } x \in [a, b] \setminus X^*. \end{cases} \tag{5}$$

Write

$$\begin{cases} X_+ = \{x \in \mathscr{X} : f(x) - p(x) = \| f - p \| \}, \\ X_- = \{x \in \mathscr{X} : f(x) - p(x) = -\| f - p \| \}, \\ \quad X = X_+ \cup X_-, \end{cases}$$

and

$$\begin{cases} X'_+ = X_l \backslash X^*, \\ X'_- = X_u \backslash X^*, \\ \quad X' = X'_+ \cup X'_-, \end{cases}$$

where

$$X_l = \{x \in [a, b] : d(p(x), l) = 0\},$$
$$X_u = \{x \in [a, b] : d(p(x), u) = 0\}.$$

Clearly, both X_l and X_u are closed sets because p is continuous. For $x \in X \cup X'$, let

$$\sigma(x) = \begin{cases} 1, & x \in X_+ \cup X'_+, \\ -1, & x \in X_- \cup X'_-, \end{cases}$$

and for $x \in X^*$, let

$$\sigma(x) = \begin{cases} -\nu(-1)^{(\mu-1)t(x)/2}, & \\ \qquad \text{if there exist } \mu, \nu \text{ such that } t_{\mu,\nu}(x) > t(x), & \\ 0, \qquad \text{otherwise} & \end{cases} \tag{6}$$

(by the definition of $t(x)$ it is not difficult to check that $\sigma(x)$ is a single-valued function on X^*). Moreover, write

$$X''_+ = \{x \in X^* : \sigma(x) = 1\},$$
$$X''_- = \{x \in X^* : \sigma(x) = -1\},$$
$$X'' = X''_+ \cup X''_-.$$

DEFINITION. $x_1, ..., x_m$ are said to be *m alternating points* of p with respect to f and l, u, if $a \leqslant x_1 < \cdots < x_m \leqslant b$, each $x_i \in X \cup X' \cup X''$, and

$$(-1)^{\tau(x_i)} \sigma(x_i) = (-1)^{i-1} (-1)^{\tau(x_1)} \sigma(x_1), \qquad i = 2, ..., m,$$

where

$$\tau(x) = \sum_{\xi \in [a, x] \cap X^*} t(\xi).$$

Let

$$T = \max\{t(x) : x \in [a, b]\}.$$

Provided $T \leqslant r$ and $t(x) < r$, $x \in X''$, we call $t(x)$ the *order of quasi-touch of l and u at* x, and T the *order of quasi-touch of l and u on* $[a, b]$ (in the next section we see that the values of $t(x)$ and T are in fact independent of the choice of $p \in K$). And by (15) below we find that when (1) holds $\sum_{x \in X^*} t(x) < n$ (and thus X^* is a finite set). So if we let

$$\Psi = \{q \in \Phi_n : q^{(0)}(x) = \cdots = q^{(t(x)-1)}(x) = 0, x \in X^*\},$$

$$n_p = n - \sum_{x \in X^*} t(x),$$

then the dimension of the subspace Ψ is n_p and we can denote by $\psi_1, ..., \psi_{n_p}$ a basis of Ψ.

Now, if

$$(X_+ \cap X_u) \cup (X_- \cap X_l) \neq \varnothing, \tag{7}$$

then p is clearly a best approximation to f from K. Otherwise, we have the following

CHARACTERIZATION THEOREM. *Assume that $\mathscr{X} \subset [a, b]$ is a compact set consisting of at least $n+1$ points, Φ_n is an extended Haar space of order r $(1 \leqslant r \leqslant n)$ on $[a, b]$, and $p \in K \subset \Phi_n$, where K is a set of generalized polynomials having restricted ranges with $T \leqslant r$ and $t(x) < r$, $x \in X''$. If $K \setminus \{p\} \neq \varnothing$, $f \in C(\mathscr{X}) \setminus K$, and (7) is false, then the following statements are equivalent*:

(i) *p is a best approximation to f from K;*

(ii) *the origin of the subspace Ψ belongs to the convex hull of the set $\{\sigma(x)(\psi_1^{(t(x))}(x), ..., \psi_{n_p}^{(t(x))}(x)) : x \in X \cup X' \cup X''\}$;*

(iii) *there exist on $[a, b]$ at least $n_p + 1$ alternating points of p with respect to f and l, u.*

Note 1. The Characterization Theorem is a general one with very weak assumptions. All of the results on characterization in [7–10] are special cases of it. In fact, the result of [7] is only a special case of Theorem 3.2 in [10] with the set of nodes being empty. However, if we apply the Characterization Theorem in the situation of [10], it follows that $t(x) = 0$, $x \in X \cup X'$; $t(x) = 1$, $x \in X^*$; $T = r = 1$; and $X'' = \varnothing$. So (ii) of the Characterization Theorem becomes $0 \in \text{co}(\{\sigma(x)(\psi_1(x), ..., \psi_{n_p}(x)) : x \in X \cup X'\})$, which is exactly (b) of Theorem 3.2 in [10], and (iii) becomes there exist $n+1$ points $\xi_1 < \cdots < \xi_{n+1}$ in $X \cup X' \cup X^*$ such that $\sigma(\xi_j) = (-1)^{j+1}\sigma(\xi_1)$, $j = 2, ..., n+1$ holds if we reset $\sigma(\xi_j) = 1$ or -1 properly for each $\xi_j \in X^*$, which is just (c) of Theorem 3.2. And under the conditions of Theorem 1 in [8], it follows that $t(x) = 0$, $x \in X \cup X'$, and $X'' = \emptyset$ again. Then (ii) is exactly (b) there. Since n_p and $(-1)^{\tau(x_i)}$ are just $n-m$ and $(-1)^m \pi(x_i)$ in [8], respectively, (iii) coincide with (c) there. Moreover, under the hypotheses of the alternation theorem in [9], it is easy to check that for each $x_i \in X^*$, $t(x_i)$ is even and hence $\tau(x_i) \equiv 1$. So (iii) becomes there exist $x_0 < x_1 < \cdots < x_{n_p}$ in $[a, b]$ which are elements of $X_+ \cup X'_+ \cup X''_+$ and $X_- \cup X'_- \cup X''_-$ alternately. This is exactly the alternation criterion given by [9].

Note 2. The Characterization Theorem is applicable to many standard restricted approximations including interpolatory constrained approximation, one-sided approximation, and copositive approximation. It is worth while to describe the copositive case in detail. First, in 1977 Passow and Taylor [4] developed a characterization in the form of convex hull and alternation under some strong conditions; second Y. K. Shih [5] gave a criterion for p to be a best copositive approximation provided that $p'(x_i) \neq 0$ at each point x_i where $f(x)$ alters its signs; at last, in 1988 J. Zhong [6] solved the general case removing Shih's additional condition. However, when we apply the Characterization Theorem in the situation we find easily that $t(x) = 0$, $x \in X \cup X'$ and $t(x) = 1$, $x \in X^*$. Then (ii) has the form of $0 \in co(\{\sigma(x)\ (\psi_1((x), ..., \psi_{n_p}(x)) : x \in X \cup X'\} \cup \{\sigma(x)(\psi'_1(x), ..., \psi'_{n_p}(x)) : x \in X''\})$, which is just the convex hull criterion given by [6]. In addition, it is not difficult to rewrite uniformly the concept of alternating k times on k intervals given by [6] as $k+1$ alternating points. Then the alternation criterion in [6] coincides with (iii) here.

3. LEMMAS

For $q \in \Phi_n$ and $x \in (a, b)$, we can find a positive number δ small enough such that q is sign-preserving in the right δ-neighbourhood and the left δ-neighbourhood of x. So we can define

$$\begin{cases} R(x, q) = \operatorname{sgn} q(\xi), & x < \xi < x + \delta, \\ L(x, q) = \operatorname{sgn} q(\xi), & x - \delta < \xi < x, \end{cases}$$

which are called the *right-sign* and *left-sign* of q at x, respectively. Write them as $R(x)$, $L(x)$ or R, L briefly when this will not lead to misunderstanding. Similarly we can define $R = R(a) = R(a, q)$ and $L = L(b) = L(b, q)$.

Obviously, if x is a t-fold zero of q with $t<r$, then by Taylor's formula we can get

$$L(x)=(-1)^t R(x). \tag{8}$$

We call x a *singular zero* of q if $x \in (a, b)$ is an r-fold zero of q and

$$L(x)=-(-1)^r R(x).$$

LEMMA 1. *Assume $x \in X^*$ is a zero of order t of $q \in \Phi_n$.*

(i) *If $1 \leqslant t < r$, then there exists a positive number λ_0 such that*

$$l(\xi) \leqslant p(\xi)+\lambda q(\xi) \leqslant u(\xi), \qquad \forall 0<\lambda<\lambda_0. \tag{9}$$

holds in a certain right neighbourhood and left neighbourhood of x if and only if

$$t_{1,R}(x) \leqslant t$$

and

$$t_{-1,L}(x) \leqslant t$$

respectively, where $R=R(x, q)$, $L=L(x, q)$.

(ii) *If $t \geqslant t(x)$, and $t(x)<r$ and $\sigma(x)\, q^{(t(x))}(x)>0$ when $x \in X''$, then there exists a $\lambda_0>0$ such that (9) holds in a certain neighbourhood of x.*

Proof. (i) We prove the lemma only in a right neighbourhood of x; the proof in the other case is similar.

Sufficiency. Assume that $R=1$. Based on the definition of $\tau_{1,1}(x)$, there exist positive numbers ε and $\delta<1$ such that for any $\xi \in (x, x+\delta)$ we have

$$u(\xi)-p(\xi) \geqslant \varepsilon\, |\xi-x|^{\tau_{1,1}(x)}.$$

And it might be assumed that

$$q(\xi)>0, \qquad x \in (x, x+\delta).$$

By Taylor's formula,

$$q(\xi)=\frac{1}{t!} q^{(t)}(\xi')(\xi-x)^t, \qquad x<\xi'<\xi. \tag{10}$$

Then from the continuity of $q^{(t)}$ and $\tau_{1,1}(x) \leqslant t_{1,1}(x) \leqslant t$ there exists a $\lambda_0>0$ such that for any $0<\lambda<\lambda_0$ we have

$$u(\xi)-p(\xi) \geqslant \lambda q(\xi), \qquad \forall \xi \in (x,\ x+\delta).$$

So (9) holds in $(x, x+\delta)$.

In the case $R=-1$, it can be proved analogously.

Note. The conclusion is still true if $t=r$. Indeed, in the case $R=1$, by the fact that $\tau_{1,1}(x)=r$ implies $t_{1,1}(x)>r=t$ we can get $\tau_{1,1}(x)\leqslant r-1=t-1$. So substituting (10) by

$$q(\xi)=\frac{1}{(t-1)!}\,q^{(t-1)}(\xi')(\xi-x)^{t-1}, \qquad x<\xi'<\xi$$

we can prove (9) for $\xi\in(x,x+\delta)$ in the same way.

Necessity. Let $t_{1,R}(x)>t$. By the definition of $t_{1,R}(x)$ it is easy to check that

$$\tau_{1,R}(x)>t. \tag{11}$$

For any right neighbourhood of x, there must be a subinterval $(x,x+\delta)$ such that $q(\xi)$ preserves the same sign in $(x,x+\delta)$ and

$$\eta=\min_{\xi'\in(x,x+\delta)}\frac{1}{t!}\,|q^{(t)}(\xi')|>0.$$

So by (10) we have

$$|q(\xi)|\geqslant\eta\,|\xi-x|^t, \qquad \forall x<\xi<x+\delta. \tag{12}$$

If $R=1$, then for any $\lambda>0$ by the definition of $\tau_{1,1}(x)$ there exists $\xi_1\in(x,x+\delta)$ such that

$$u(\xi_1)-p(\xi_1)<\lambda\eta\,|\xi_1-x|^{\tau_{1,1}(x)-1}.$$

Combined with (11) and (12) we have

$$u(\xi_1)-p(\xi_1)<\lambda\,|q(\xi_1)|=\lambda q(\xi_1).$$

So (9) is false for any $\lambda_0>0$ and any right neighbourhood of x. The proof in the case $R=-1$ is similar.

(ii) Presume that $x\in(a,b)$ (it can be proved similarly if $x=a$ or b). If $x\bar{\in}X''$, then by $\sigma(x)=0$ and (6) we have

$$t_{1,R}(x),\ t_{-1,L}(x)\leqslant t(x). \tag{13}$$

And from (i) and the note in its proof of sufficiency we get the conclusion needed.

Now assume that $x\in X''$. Then $\sigma(x)\,q^{(t(x))}(x)>0$ and $t=t(x)<r$. In the case $t(x)=t_\pm(x)$, we have

$$\begin{aligned} R&=\operatorname{sgn} q^{(t(x))}(x)=\sigma(x)=-\nu\omega^{(\mu-1)/2},\\ L&=(-1)^{t(x)}R=-\nu\omega^{(\mu+1)/2} \end{aligned} \tag{14}$$

provided

$$t_{\mu,\nu}(x) > t(x).$$

If $\mu = 1$, then by the definition of $t_{\pm}(x)$ and $t(x)$ it follows that

$$t_{1,-\nu}(x),\ t_{-1,-\omega\nu}(x) \leqslant t(x).$$

So we can get (13) from (14). If $\mu = -1$, we can get (13) just the same.

If $t(x) = t_{\pm}(x) + 1$, then there exists ν such that

$$t_{1,\nu}(x),\ t_{-1,-\omega\nu}(x) > t_{\pm}(x).$$

So by the definition of $t_{\pm}(x)$ we have

$$t_{1,-\nu}(x),\ t_{-1,\omega\nu}(x) \leqslant t_{\pm}(x) < t(x),$$

and hence the condition in (6) is

$$t_{1,\nu}(x) > t(x) \qquad \text{or} \qquad t_{-1,-\omega\nu}(x) > t(x).$$

In both cases, similar to (14) it follows that

$$R = -\nu \qquad \text{and} \qquad L = \omega\nu.$$

Then (13) holds again.

The lemma is proved.

LEMMA 2. *Assume that $T \leqslant r$ and $t(x) < r$, $x \in X''$. If $p + q \in K$, then*

$$q^{(0)}(x) = \cdots = q^{(t(x)-1)}(x) = 0, \qquad x \in X^*, \tag{15}$$

and

$$\sigma(x)\, q^{(t(x))}(x) \geqslant 0, \qquad x \in X' \cup X''. \tag{16}$$

Proof. If there exists $x \in X'$ for which (16) does not hold, then from (5) we get $t(x) = 0$ and by the definition of $\sigma(x)$ it follows that $p + q \bar{\in} K$. This is impossible.

If there exists $x \in X^*$ for which (15) does not hold, then there exists a positive integer $t < t(x)$ such that x is a zero of order t of q. By the definition of $t_{\pm}(x)$, we can find a μ such that

$$t_{\mu,1}(x),\ t_{\mu,-1}(x) \geqslant t_{\pm}(x).$$

On the basis of Lemma 1(i), we have $t_{\mu,1}(x) \leqslant t$ or $t_{\mu,-1}(x) \leqslant t$ because $p + q \in K$ and K is a convex set. So $t \geqslant t_{\pm}(x)$, and by (4) we have $t(x) = t + 1 = t_{\pm}(x) + 1$ and

$$t_{1,\nu}(x),\ t_{-1,-\omega\nu}(x) > t_{\pm}(x).$$

Hence $t_{1,R}(x)>t$ if $v=R$, and $t_{-1,\omega R}(x)=t_{-1,L}(x)>t$ if $v=-R$. Then from Lemma 1(i) it follows that $p+q\bar{\in}K$ again.

Now suppose there exists $x\in X''$ for which (15) holds but (16) does not. Then x is a zero of order $t(x)<r$ of q. If in (6) $t_{1,v}(x)>t(x)$ holds, then $v=-\sigma(x)=R$ and $t_{1,R}(x)>t(x)$, and if $t_{-1,v}(x)>t(x)$ then $v=-(-1)^{t(x)}\sigma(x)=L$ and $t_{-1,L}(x)>t(x)$. Hence we have $p+q\bar{\in}K$ again by Lemma 1(i).

The lemma is established.

LEMMA 3. *If* $a=x_1<x_2<\cdots<x_{k-1}<x_k=b$ $(k\geqslant 2)$, $\tau=\sum_{i=1}^k t_i\leqslant n-1$ *with* $0\leqslant t_i\leqslant r$, *and*

$$n-1-\tau \text{ is even}, \qquad \text{if } t_1 \text{ or } t_k=r, \tag{17}$$

then there exists a $q\in\Phi_n$ *such that* x_i *is its zero of order* t_i $(i=1,...,k)$ *and*

$$(-1)^{t_1+\cdots+t_i}q(x)>0, \qquad x\in(x_i,x_{i+1}),\ i=1,...,k-1.$$

Proof. Write

$$k_0=\left[\frac{n-1-\tau}{2}\right], \qquad t_0=n-1-\tau-2k_0.$$

By the hypotheses it is easy to check that $t_j+t_0\leqslant r$, $j=1,k$. Let X_1 and X_k be two arbitrary subsets of $[a,b]\backslash\{x_i\}_{i=1}^k$ each consisting of k_0 points, and $X_1\cap X_k=\varnothing$. On the basis of the definition of an extended Chebyshev system and Theorem 5.2 in [1, Chap. 1], for $j=1$ and k we can find $q_j\in\Phi_n$ for which x_i $(i\neq j)$ is its t_i-fold zero, t_j is its t_j+t_0-fold zero, and each point of X_j is its 2-fold zero (or nonnodal zero if $r=1$). Then multiplied by -1 if necessary (denoted by q_j still) it follows that

$$(-1)^{t_1+\cdots+t_i}q_j(x)>0, \qquad x\in(x_i,x_{i+1})\backslash X_j,\ i=1,...,k-1.$$

Now $q=q_1+q_k$ meets the requirements of the lemma.

LEMMA 4. *Assume that* $q\in\Phi_n$ *has a total of* $\tau=\sum_{i=1}^k t_i$ *zeros counting multiplicities, and* x_i *is its zero of order* $t_i\geqslant 1$, $i=1,...,k$. *If by* $|I|$ *we denote the number of the elements of the set*

$$I:=\{i:x_i \text{ is a singular zero of } q\},$$

then

$$\tau+|I|\leqslant n-1. \tag{18}$$

Proof. It is sufficient to give a proof provided $|I|>0$. Let

$$I_0=\{i:x_i\bar{\in}(a,b),\ t_i=r\}.$$

By Lemma 3 we can make a generalized polynomial q_1 such that x_i $(i \in I \cup I_0)$ is its zero of order $r-1$, $x_i (i \bar{\in} I \cup I_0)$ is its zero of order t_i, and q_1 has the same signs as q on $[a, b] \setminus \{x_i : i \in I \cup I_0\}$. Thus, $q_\lambda = q - \lambda q_1 \not\equiv 0$ has at least $\tau - |I| - |I_0|$ zeros. Take a $\delta > 0$ sufficiently small so that

$$(x_j - \delta, x_j + \delta) \cap (\{x_i\}_{i=1}^k \cup \{a, b\}) = \{x_j\}, \qquad j = 1, ..., k.$$

For $x_j, j \in I$ or $x_j = a, j \in I_0$, we have

$$\operatorname{sgn} q_\lambda (x_j + \delta) = \operatorname{sgn} q(x_j + \delta)$$

if $\lambda > 0$ is sufficiently small. Since for any $\eta > 0$ we have

$$q_\lambda(x_j + \eta) = \frac{\eta^{r-1}}{(r-1)!} [q^{(r-1)}(\xi) - \lambda q_1^{(r-1)}(\xi)], \qquad x_j < \xi < x_j + \eta,$$

therefore $q^{(r-1)}(x_j) = 0$ and $q_1^{(r-1)}(x_j) \neq 0$ imply

$$\operatorname{sgn} q_\lambda(x_j + \eta) = -\operatorname{sgn} q(x_j + \eta)$$

if $\eta < \delta$ is sufficiently small. So there exists a zero of q_λ in $(x_j + \eta,\ x_j + \delta)$. In the same manner, for $x_j, j \in I$ or $x_j = b, j \in I_0$, we can find $0 < \eta' < \delta$ such that there exists a zero of q_λ in $(x_j - \delta,\ x_j - \eta')$. Thus, if λ is sufficiently small, then q_λ has at least $(\tau - |I| - |I_0|) + 2|I| + |I_0| = \tau + |I|$ zeros, which implies (18).

LEMMA 5. *Assume $T \leqslant r$ and $t(x) < r$, $x \in X''$. Then*

(i) *there exists a $q_0 \in \Phi_n$ such that each $x \in X^*$ is its zero of order $t(x)$, and*

$$\sigma(x)\, q_0^{(t(x))}(x) > 0, \qquad \forall x \in X' \cup X''; \tag{19}$$

(ii) *for $q_0 \in \Phi_n$, if each $x \in X^*$ is its zero of order at least $t(x)$, and* (19) *holds, then there exists $\lambda_0 > 0$ such that $p + \lambda q_0 \in K$, $0 < \lambda \leqslant \lambda_0$.*

Proof. (i) On the basis of (1), we can find a $q \not\equiv 0$ such that $p + q \in K$. Assume that the zeros of q in X' are $x_1, ..., x_{i_1}$; the zeros in X^* are $x_{i_1+1}, ..., x_{i_2}$ (clearly $\{x_i\}_{i=i_1+1}^{i_2} = X^*$); and the zeros which do not belong to the set $X' \cup X^*$ are $x_{i_2+1}, ..., x_{i_3}$. And assume each x_i is a zero of order $t_i \geqslant 1$, $i = 1, ..., i_3$.

Write the right-sign and left-sign of q at x_i as $R_i = R(x_i, q)$ and $L_i = L(x_i, q)$, respectively (note that only one of them is defined if $x_i = a$ or b). By ρ_0 we denote the minimum of the distance between any two different points in $\{a, b\} \cup \{x_i\}_{i=1}^{i_3}$. And we define I and I_0 the same as in Lemma 4.

In which follows, we choose t_i' $(i = 1, ..., i_3)$ points (at most r points are coincident with each other) in a certain closed interval $F_i \subset [x_i - \rho_0/3, x_i + \rho_0/3]$ with t_i' satisfying

$$\begin{cases} t_i - t'_i = \text{nonnegative integer}, & \text{if } x_i \bar{\in} (a, b), \\ t_i - t'_i = \text{nonnegative even integer}, & \text{if } x_i \in (a, b) \text{ and } i \bar{\in} I, \\ t_i + 1 - t'_i = \text{nonnegative even integer}, & \text{if } x_i \in (a, b) \text{ and } i \in I. \end{cases} \tag{20}$$

Since Lemma 4 implies $\sum_{i=1}^{i_3} t'_i \leqslant n-1$, according to Lemma 3 we can find a q_0 with these $\sum_{i=1}^{i_3} t'_i$ zeros chosen (if a or b is selected to be a zero of order r, we add a single zero if necessary so that (17) holds). Then we prove that q_0 satisfies the requirement of the lemma.

For $1 \leqslant i \leqslant i_1$, if $x_i \in X'_+$ (or X'_-), let ρ_i be the distance between x_i and the closed set X_u (or X_l) (if X_u (or X_l) is empty, let $\rho_i = +\infty$). Writing

$$\rho = \min\{\rho_i : i = 0, 1, ..., i_1\},$$

we have $\rho > 0$. And if $x_i = a$ or b, we define $L_i = R_i$ or $R_i = L_i$ in addition. Let

$$F_i = \begin{cases} \left[x_i - \frac{\rho}{3}, x_i + \frac{\rho}{3}\right] \cap [a, b], & \text{if } L_i = R_i \text{ and } R_i\sigma(x_i) < 0, \\ \{x_i\}, & \text{if } L_i = R_i \text{ and } R_i\sigma(x_i) > 0, \\ \left[x_i, x_i + \frac{\rho}{3}\right], & \text{if } L_i \neq R_i \text{ and } R_i\sigma(x_i) < 0, \\ \left[x_i - \frac{\rho}{3}, x_i\right], & \text{if } L_i \neq R_i \text{ and } R_i\sigma(x_i) > 0. \end{cases} \tag{21}$$

By t'_i we denote the number of the endpoints of F_i not being x_i. It is easy to check that t'_i satisfies (20). And from the definition of ρ we can see that

$$\sigma(x) = \sigma(x_i), \qquad \forall x \in X' \cap F_i. \tag{22}$$

If $i_1 < i \leqslant i_2$, let

$$F_i = \begin{cases} [x_i - \rho', x_i + \rho'], \\ \qquad \text{if } R_i = -\sigma(x_i) \text{ and } L_i = -(-1)^{t(x_i)}\sigma(x_i), \\ [x_i, x_i + \rho'], \\ \qquad \text{if } R_i = -\sigma(x_i), \text{ and } x_i = a \text{ or } L_i = (-1)^{t(x_i)}\sigma(x_i), \\ [x_i - \rho', x_i], \\ \qquad \text{if } L_i = -(-1)^{t(x_i)}\sigma(x_i), \text{ and } x_i = b \text{ or } R_i = \sigma(x_i), \\ [x_i, x_i + \rho'], \\ \qquad \text{if } \sigma(x_i) = 0, L_i \neq (-1)^{t(x_i)} R_i, \\ \{x_i\}, \quad \text{otherwise}, \end{cases} \tag{23}$$

where $\rho' < \rho_0/3$ is a positive number such that

$$X' \cap F_i = \varnothing, \qquad i_1 < i \leqslant i_2. \tag{24}$$

The existence of ρ' can be proved as follows. For fixed $i \in \{i_1+1, ..., i_2\}$, if $\sigma(x) \neq 0$ then by the hypothesis we can get $t(x_i) < r$ and find a generalized polynomial $\bar{q}$ having a zero of order $t(x_i)$ at x_i such that $\sigma(x_i)\,\bar{q}^{(t(x_i))}(x_i) > 0$. According to Lemma 1(ii), there exist $\lambda > 0$ and $\rho' > 0$ such that

$$l(x) \leqslant p(x) + \lambda\bar{q}(x) \leqslant u(x), \qquad \forall x \in [x_i - \rho', x_i + \rho'] \cap [a, b].$$

In the first and second cases of (23), when $\rho' < \rho_0/3$ is suficiently small, we have

$$\operatorname{sgn}[\lambda\bar{q}(x)] = R(x_i, \bar{q}) = \operatorname{sgn} \bar{q}^{(t(x_i))}(x_i) = \sigma(x_i) = -R_i = -\operatorname{sgn} q(x),$$
$$\forall\, x \in (x_i, x_i + \rho'] \cap [a, b].$$

And in the first and third cases

$$\operatorname{sgn}\,[\lambda\bar{q}(x)] = L(x_i, q) = (-1)^{t(x_i)} \operatorname{sgn} \bar{q}^{(t(x_i))}(x_i) = (-1)^{t(x_i)}\, \sigma(x_i)$$
$$= -L_i = -\operatorname{sgn} q(x), \qquad \forall x \in [x_i - \rho', x_i) \cap [a, b].$$

Then in cases 1–3, for any $x \neq x_i$ in F_i from $l(x) \leqslant p(x) + q(x) \leqslant u(x)$ we can get $d(p(x), l) > 0$ and $d(p(x), u) > 0$. So (24) holds. And under the conditions in the fourth case of (23) we can prove similarly that $X' \cap F_i = \varnothing$ provided $\bar{q}$ has a zero of order $t(x_i)$ at x_i and $-R_i R(x_i, \bar{q}) > 0$.

Note. In fact, if $\sigma(x_i) = 0$, which is a part of the conditions of the fourth case only, a similar argument leads to the fact that, for $F_i = [x_i, x_i + \rho']$ (if $x_i \neq b$) or $F_i = [x_i - \rho', x_i]$ (if $x_i \neq a$) it still follows that $F_i \cap X' = \varnothing$ (in the latter case it is required that $-L_i L(x_i, \bar{q}) > 0$).

Let t_i' be the sum of $t(x_i)$ and the number of the endpoints of F_i not being x_i. Let us prove that t_i' satisfies (20). Before all, by Lemma 2 we have

$$t_i - t(x_i) \geqslant 0. \tag{25}$$

In the first case of (23), clearly $x_i \in (a, b)$ and $L_i = (-1)^{t(x_i)} R_i$. (a) If $i \in I$, then by

$$L_i \neq (-1)^{t_i} R_i \tag{26}$$

and (25) we see that $t_i - t(x_i)$ is a positive odd number. So the third expression of (20) holds. (b) If $i \bar{\in} I$, then $t_i - t(x_i)$ is a nonnegative even number because (26) is false. Provided $t_i = t(x_i)$, by $\sigma(x_i) \neq 0$ we get $t_i < r$. If $t_{1,\nu}(x_i) > t(x_i)$ holds in (6), then $\nu = -\sigma(x_i)$ and $t_{1,R_i}(x_i) > t_i$, and by

Lemma 1(i) we get $p+q \bar{\in} K$; if $t_{-1,v}(x_i) > t(x_i)$, then $v = -(-1)^{t(x_i)} \sigma(x_i)$ and $t_{-1,L_i}(x_i) > t_i$, and we get a contradiction again. Thus the second expression of (20) holds. In the second to fourth cases of (23), if $x \in (a, b)$, by $L_i \neq (-1)^{t(x_i)} R_i$ it can be found that $t_i - t(x_i)$ is a nonnegative even number or a positive odd number if $i \in I$ or $i \bar{\in} I$ respectively. So we get the third or second expression of (20). If $x_i = a$ or b, we can find a contradiction similar to (b) in the first case provided $t_i - t(x_i) = 0$. thus we get the first expression of (20). In the fifth case, (25) implies the first expression of (20) if $x_i \bar{\in} (a, b)$; and if $x_i \in (a, b)$, by $L_i = (-1)^{t(x_i)} R_i$ the second or third expression of (20) can be gotten.

Again, for $i_2 < i \leqslant i_3$, let

$$F_i = \{x_i\}.$$

Clearly we have

$$X' \cap F_i = \emptyset, \qquad i_2 < i \leqslant i_3. \tag{27}$$

Let $t_i' = t_i - 1$ if $i \in I \cup I_0$ and $t_i' = t_i$ if $i \bar{\in} I \cup I_0$; then (20) holds.

Now, according to Lemma 3, we make a generalized polynomial q_0 having $\sum_{i=1}^{i_3} t_i'$ zeros such that the endpoints of F_i not being x_i are its single zeros, each point in $\{x_i\}_{i=i_1+1}^{i_2} = X^*$ is its zero of order $t(x_i)$, and x_i ($i = i_2 + 1, ..., i_3$) is its zero of order t_i'. In doing this, (17) is provided as it should be. Otherwise, $n - 1 - \sum_{i=1}^{i_3} t_i'$ is odd and a or b is selected to be an r-fold zero. Since by the selection of the zeros there exists an $x_i = a$ or b such that $x_i \in X^*$ and $t(x_i) = r$, we have $x_i \bar{\in} X''$, which means $\sigma(x_i) = 0$, and $F_i = \{x_i\}$ by (23). So if we reset

$$F_i = \begin{cases} [x_i, x_i + \rho'], & \text{if } x_i = a, \\ [x_i - \rho', x_i], & \text{if } x_i = b \end{cases}$$

with (24) remaining true because of the Note below the proof of (24), then $n - 1 - \sum_{i=1}^{i_3} t_i'$ becomes a nonnegative even number and we can get q_0 by Lemma 3. By the definition of F_i and t_i', we see that q_0 (multiplied by -1 if necessary) has the same signs as q on $[a, b] \backslash (\bigcup_{i=1}^{i_3} F_i)$. From Lemma 2 we see that $\sigma(x) q^{(t(x))}(x) > 0$, $x \in X' \backslash (\bigcup_{i=1}^{i_3} F_i)$. So by $t(x) = 0$ we get

$$\sigma(x) q_0^{(t(x))}(x) > 0, \qquad x \in X' \Big\backslash \left(\bigcup_{i=1}^{i_3} F_i \right).$$

By (23), our selection of the zeros ensures that, for $x \in X''$ $R(x, q_0) = \sigma(x)$ (or $L(x, q_0) = (-1)^{t(x)} \sigma(x)$ if $x = b$). So

$$\sigma(x) q_0^{(t(x))}(x) > 0, \qquad x \in X''.$$

In addition, if $x \in F_i$, $1 \leqslant i \leqslant i_1$, according to (21) and the construction of q_0, it can be found that

$$\sigma(x_i)[q_0(x) - q(x)] > 0.$$

So when $x \in X' \cap F_i$, from (22) and Lemma 2 we have

$$\sigma(x)\, q_0^{(t(x))}(x) > \sigma(x)\, q(x) \geqslant 0.$$

Considering (24) and (27), the conclusion of (i) is gotten.

(ii) By $O(S, \delta)$ we denote a δ-neighbourhood of a set $S \subset [a, b]$. From Lemma 1(ii), there exist $\delta_1 > 0$ and $\lambda_1 > 0$ such that for any $0 < \lambda < \lambda_1$ and $x \in O(X^*; \delta_1)$ we have

$$l(x) \leqslant p(x) + \lambda q_0(x) \leqslant u(x). \tag{28}$$

By δ'(or δ'') we denote the distance between the closed sets $X'_+ \backslash O(X^*; \delta_1)$ and X_u(or $X'_- \backslash O(X^*; \delta_1)$ and X_l). Then both δ' and δ'' are positive. In addition, since $\sigma(x)\, q_0(x) > 0$ for any $x \in X' \backslash O(X^*; \delta_1)$, then δ^*, the distance between $X' \backslash O(X^*; \delta_1)$ and the set of the zeros of q_0 is also positive. Letting $\delta_2 = \min\{\delta'/2, \delta''/2, \delta^*\}$, because for any $x \in O(X'_+ \backslash O(X^*; \delta_1); \delta_2)$ we have $q_0(x) > 0$, therefore

$$l(x) < p(x) + \lambda q_0(x),$$

and

$$u(x) - p(x) \geqslant \inf_{\xi \in [a, b] \backslash O(X_u; \delta_2)} d(p(\xi), u) \tag{29}$$

since $x \bar{\in} O(X_u; \delta_2)$. On account of the infimum in (29) being a positive constant, (28) holds for λ sufficiently small. Discussing similarly $x \in O(X'_- \backslash O(X^*; \delta_1); \delta_2)$ we can find a positive number $\lambda_2 < \lambda_1$ such that (28) holds for any $0 < \lambda < \lambda_2$ and $x \in O(X' \backslash O(X^*; \delta_1); \delta_2)$.

By F we denote the closed set $[a, b] \backslash [O(X^*; \delta_1) \cup O(X' \backslash O(X^*; \delta_1); \delta_2)]$. Then $F \cap X_l = F \cap X_u = \varnothing$ and

$$\inf_{x \in F} [p(x) - l(x)] > 0, \qquad \inf_{x \in F} [u(x) - p(x)] > 0.$$

So there exists a positive number $\lambda_0 < \lambda_2$ such that, when $0 < \lambda < \lambda_0$ (28) holds for any $x \in F$, and hence for any $x \in [a, b]$.

The lemma is established.

Before the end of this section, we give a new explanation for the order of quasi-touch $t(x)$. If $T \leqslant r$ and $t(r) < r$, $x \in X''$, then for any $q_1, q_2 \in K$, by Lemma 2 we see that each $x \in X^*$ is a zero of order at least $t(x)$ of $q_1 - p$

and $q_2 - p$, and consequently of $q_1 - q_2$. And for $x \bar{\in} X^*$, x is clearly a zero of order at least $t(x)$ of $q_1 - q_2$ because $t(x) = 0$. On the other hand, if we let $q_1 = p + \lambda q_0 \in K$, where q_0 satisfies the conditions of Lemma 5(i), and $q_2 = p$, then each $x \in X^*$ is a zero of order $t(x)$ of $q_1 - q_2 = \lambda q_0$. And for each $x \bar{\in} X^*$ subject to $q_0(x) = 0$, if in making q_0 in the proof of Lemma 5(i) we shift this zero to the right or left slightly then we can get q_0 satisfying the requirement of Lemma 5(i) with $q_0(x) \neq 0$. So there exist $q_1, q_2 \in K$ such that x is a zero of order $t(x) = 0$ of $q_1 - q_2$. Now we see that $t(x)$ is just the minimum of the order of the zero x of $q_1 - q_2$ for all choices of $q_1, q_2 \in K$. Therefore, though $t(x)$ was defined by the given generalized polynomial $p \in K$, in fact it is independent on the choice of p in K, but depends only on the values of 1 and u.

4. Proof of Characterization Theorem

Proof of Characterization Theorem. First, based on the presumption that (7) is false we see that $\sigma(x)$ has at most one value for any $x \in [a, b]$. Next, it is easy to prove that $X \cup X' \cup X''$ is compact. In fact, X and X'' are closed sets clearly. Provided $\xi_i \in X'_+$, $\xi_i \to x$ $(i \to \infty)$, and $x \bar{\in} X'_+$, because X_t is closed and $X'_+ = X_t \backslash X^*$, therefore $x \in X^*$ and there exists a μ such that $t_{\mu, -1}(x) = \tau_{\mu, -1}(x) = +\infty$. So by (6) we see that $x \in X''$. Dealing with the limit points of X'_- similarly we can see that $X' \cup X''$ is a compact set.

(i) $\Rightarrow$ (ii) If (ii) is false, then by the Linear Inequality Theorem (see [11, Chap. 1, Sect. 5) there exists a $q_0 \in \Psi$ such that

$$\sigma(x)\, q_0^{(t(x))}(x) > 0, \qquad \forall x \in X \cup X' \cup X''. \tag{30}$$

So from Lemma 5(ii) we can find a $\lambda_0 > 0$ such that $p + \lambda q_0 \in K$ for any $0 < \lambda < \lambda_0$.

Since the failure of (7) implies $X \cap X^* = \varnothing$, for any $x \in X$ we have $t(x) = 0$, and by (30) it follows that

$$\operatorname{sgn} q_0(x) = \operatorname{sgn}[f(x) - p(x)]. \tag{31}$$

Because X is closed, by the continuity of the functions we can find a $\delta > 0$ such that both q_0 and $f - p$ are sign-preserving in the δ-neighbourhood of any $x \in X$ and

$$\eta = \inf_{x \in O(X;\, \delta)} |q_0(x)| > 0,$$

$$\inf_{x \in O(X;\, \delta) \cap \mathscr{X}} |f(x) - p(x)| > 0.$$

So (31) holds for any $x \in O(X; \delta) \cap \mathscr{X}$ and there exists a positive number $\lambda_1 \leqslant \lambda_0$ such that

$$\lambda_1 |q_0(x)| < |f(x) - p(x)|, \qquad \forall x \in O(X; \delta) \cap \mathscr{X}.$$

Therefore,

$$|f(x) - p(x) - \lambda q_0(x)| = |f(x) - p(x)| - \lambda |q_0(x)| \leqslant \|f - p\| - \lambda \eta$$

for any $0 < \lambda \leqslant \lambda_1$ and $x \in O(X; \delta) \cap \mathscr{X}$. Since on the compact set $\mathscr{X} \backslash O(x; \delta)$ we have $|f(x) - p(x)| < \|f - p\|$, then for sufficiently small λ it follows that $|f(x) - p(x) - \lambda q_0(x)| < \|f - p\|$ for any $x \in \mathscr{X} \backslash O(X; \delta)$. Hence we get $\|f - (p + \lambda q_0)\| < \|f - p\|$ on the contrary.

(ii) $\Rightarrow$ (iii) Assume that there exist at most m alternating points $y_1, ..., y_m$ of p with respect to f and l, u, but $m \leqslant n_p$. Write $y_0 = a$, $y_{m+1} = b$. For $i = 1, ..., m$, let

$$Y_i = \{ y \in [y_{i-1}, y_{i+1}] : y \in X \cup X' \cup X'', (-1)^{\tau(y)} \sigma(y) = (-1)^{\tau(y_i)} \sigma(y_i) \},$$

$$y_i' = \inf_{y \in Y_i} y, \qquad y_i'' = \sup_{y \in Y_i} y.$$

It can be proved that

$$y_i'' < y_{i+1}', \qquad i = 1, ..., m-1. \tag{32}$$

It is sufficient to prove $y_i'' \in Y_i$, $y_{i+1}' \in Y_{i+1}$, because from this we can get $y_i'' \neq y_{i+1}'$, and if $y_i'' > y_{i+1}'$ then $y_1, ..., y_i, y_{i+1}', y_i'', y_{i+1}, ..., y_m$ are $m+2$ alternating points on the contrary. Choose a monotone increasing sequence $\{\eta_j\} \subset Y_i$ such that $\eta_j \to y_i''$ $(j \to \infty)$. The compactness of $X \cup X' \cup X''$ implies $y_i'' \in X \cup X' \cup X''$. If $y_i'' \bar{\in} X''$, then there exists a positive integer J such that for $j > J$ it follows that

$$\tau(\eta_j) = \tau(y_i'')$$

and $\{\eta_j\}_{j=J+1}^{\infty} \subset X_+ \cup X_l$ (or $X_- \cup X_u$). Since X_+ and X_l (or X_- and X_u) are both closed sets, we get

$$\sigma(\eta_j) = \sigma(y_i'').$$

Thus $y_i'' \in Y_i$. Provided $y_i'' \in X''$, since $X \cap X'' = \varnothing$, then there exixts a J such that $\{\eta_j\}_{i=J+1}^{\infty} \subset X_+'$ (or X_-'). Hence $t_{-1,-1}(y_i'')$ (or $t_{-1,1}(y_i'')$) equals $+\infty$, and by (6) we have

$$\sigma(y_i'') = (-1)^{t(y_i'')} \text{ (or } -(-1)^{t(y_i'')}).$$

So there exists a j so large that

$$(-1)^{\tau(y_i'')} \sigma(y_i'') = (-1)^{\tau(\eta_j)} (-1)^{t(y_i'')} \sigma(y_i'') = (-1)^{\tau(\eta_j)} \sigma(\eta_j).$$

Thus we get $y_i'' \in Y_i$ again. Similarly we can prove $y_{i+1}' \in Y_{i+1}$. (32) is established.

For $i=1, ..., m-1$, if $(y_i'', y_{i+1}') \cap X^* = \varnothing$, then let $\xi_i = \frac{1}{2}(y_i'' + y_{i+1}')$, otherwise let $\xi_i = \frac{1}{2}(y_i'' + \min\{y : y \in (y_i'', y_{i+1}') \cap X^*\})$. By the assumptions it follows that $\tau := m-1+\sum_{x \in X^*} t(x) < n$. Rewrite the points in $\{\xi_i\}_{i=1}^{m-1} \cup X^*$ as $x_1 < \cdots < x_k$, and let

$$t_i = \begin{cases} t(x_i), & \text{if } \ x \in X^*, \\ 1, & \text{otherwise}, \end{cases} \qquad i=1, ..., k.$$

Now (a) we make a q by Lemma 3 if (17) holds. (b) When (17) is false, it might be provided that $t_1 = r$. Then we have $t(a) = r$ and hence $a \in X^* \backslash X''$ and $a < y_1'$. So if we set up an additional single zero $\frac{1}{2}(a + y_1')$ then q can be made by Lemma 3. In both cases (8) holds for each t-fold zero $x \in (a, b)$, and we can provide in addition that

$$R(y_1) = \sigma(y_1).$$

If $b \in X \cup X' \cup X''$ we define additionally $R(b) = (-1)^{t(b)} \ L(b)$. By the definition of alternating points we have

$$R(y_i) = (-1)^{\tau(y_i) - \tau(y_1) + (i-1)} \ R(y_1) = \sigma(y_i).$$

Because for any $y \in [y_i', y_i''] \cap (X \cup X' \cup X'')$ it follows that $y \in Y_i$, therefore

$$R(y) = (-1)^{\tau(y_i) - \tau(y)} \ R(y_i) = \sigma(y).$$

Since it is clear that

$$\begin{cases} (y_i'', y_{i+1}') \cap (X \cup X' \cup X'') = \varnothing \qquad i = 1, ..., m-1, \\ \{x : x < y_1'\} \cap (X \cup X' \cup X'') \{x : x > y_m''\} \cap (X \cup X' \cup X'') = \varnothing, \end{cases}$$

then we have

$$R(x) = \sigma(x), \qquad \forall x \in X \cup X' \cup X''. \tag{33}$$

When $x \in X \cup X'$, we have $t(x) = 0$ and from the construction of q we see that $q(x) \neq 0$. Hence

$$R(x) = \operatorname{sgn} q^{(t(x))}(x).$$

Since the above equality holds clearly for $x \in X''$, (33) implies that

$$\sigma(x) \, q^{(t(x))}(x) > 0, \qquad \forall x \in X \cup X' \cup X''.$$

So by the Linear Inequality Theorem we see that (ii) is false on the contrary.

(iii) $\Rightarrow$ (i) Assume that $y_1, ..., y_{n_p+1}$ are alternating points. If there exists $p+q \in K$ such that $\| f-(p+q)\| < \| f-p \|$, then

$$\sigma(x)\, q(x) > 0, \qquad \forall x \in X,$$

and

$$\min_{x \in X} |q(x)| > 0.$$

If q_1 is an generalized polynomial subject to the conditions in Lemma 5(i), then $q_0 = q + \lambda q_1$ satisfies

$$\sigma(x)\, q_0(x) > 0, \qquad \forall x \in X \tag{34}$$

for $\lambda > 0$ sufficiently small. By Lemma 2, q_0 meets (19) and each $x \in X^*$ is a zero of order at least $t(x)$ of q_0. For $i = 1, ..., n_p$ provided q_0 has just $\tau(y_{i+1}) - \tau(y_i)$ zeros and no singular zero on $(y_i, y_{i+1}]$, we have

$$R(y_{i+1}, q_0) = (-1)^{\tau(y_{i+1}) - \tau(y_i)} R(y_i, q_0) \tag{35}$$

(provided $R(b, q_0) = (-1)^{t(b)} L(b, q_0)$). If $y_i \in X$, then by (34) we get

$$R(y_i, q_0) = \operatorname{sgn} q_0(y_i) = \sigma(y_i).$$

And if $y_i \in X' \cup X''$, by (19) we get

$$R(y_i, q_0) = \operatorname{sgn} q_0^{(t(y_i))}(y_i) = \sigma(y_i).$$

Discussing y_{i+1} similarly we conclude from (35) that

$$\sigma(y_{i+1}) = (-1)^{\tau(y_{i+1}) - \tau(y_i)} \sigma(y_i),$$

which contradicts the definition of the alternating points. So q_0 has at least $\tau(y_{i+1}) - \tau(y_i) + 1$ zeros or one singular zero on each interval $(y_i, y_{i+1}]$. Then the sum of the numbers of the zeros and singular zeros is no less than

$$n_p + \sum_{x \in X^*} t(x) = n$$

which contradicts Lemma 4.

The theorem is proved.

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